JOURNAL OF APPROXIMATION THEORY **66**, 44–52 (1991)

Uniform Approximation to $|x|^\beta$ by Sinc Functions

FRITZ KEINERT

Department of Mathematics, Iowa State University, Ames, Iowa 50011

Communicated by T. J. Rivlin

Received March 21, 1990; revised September 6, 1990

The best uniform approximation to the function $|x|^\beta$, $\beta > 0$, on $[-1, 1]$ by any of the standard classes of functions of approximation theory has an asymptotic error of at best $O(e^{-c\sqrt{n}})$, where n is the dimension of the space of approximating functions. We exhibit a class of Sinc basis functions for which this error decays at the faster rate $O(e^{-cn/\log n})$, uniformly for all $0 < \beta_0 \leqslant \beta \leqslant \beta_1$. © 1991 Academic Press, Inc.

1. INTRODUCTION

One of the basic questions of approximation theory is the following: Given a function f, a norm, and a class S of approximating functions, how well can f be approximated in this norm by functions in S? This question becomes especially interesting for nonsmooth functions.

For the function $|x|^\beta$, $\beta > 0$, on the interval $[-1, 1]$, and the uniform (or Chebyshev) norm, the answer is known for many classes of functions, including the standard polynomials, piecewise polynomials, and rational functions.

In the best of these cases, the error of the best uniform approximation to $|x|^\beta$ decays like $O(e^{-c\sqrt{n}})$ as $n \to \infty$, where n is the dimension of the space of approximating functions, or more generally the number of free parameters in the class of approximating functions.

In this paper, we will exhibit a linear space of functions for which the error of the best uniform approximation to $|x|^\beta$ behaves like $O(e^{-cn/\log n})$, uniformly for all β with $0 < \beta_0 \leqslant \beta \leqslant \beta_1$. This is an interesting application of the theory of approximation with Sinc functions [10].

Section 2 briefly outlines the history of this problem and summarizes known results. In section 3, we will review the pertinent parts of Sinc approximation. Section 4, then, contains the main result of this paper, an application of Sinc methods to the problem of uniform approximation of $|x|^\beta$ on $[-1, 1]$.

0021-9045/91 $3.00

2. History and Previous Results

In 1908, de La Vallée Poussin posed the problem of estimating the error of the best uniform approximation by polynomials to the function $|x|$ on the interval $[-1, 1]$. The problem was solved by Bernstein in 1911 and published in [1]. If $E_n(f)$ denotes the error of the best approximation to f by polynomials of order n (that is, degree $(n-1)$) on $[-1, 1]$, then

$$\lim_{n \to \infty} 2nE_{2n}(|x|) = \lambda,$$

where $\lambda \approx 0.282$ is a constant. Thus,

$$E_n(|x|) = O(n^{-1}) \qquad \text{as} \quad n \to \infty.$$

The error of the best approximation of $|x|^\beta$, $0 < \beta < 1$, on $[-1, 1]$ by polynomials was also determined by Bernstein [2] to be

$$\lim_{n \to \infty} n^\beta E_n(|x|^\beta) = c(\beta),$$

where $c(\beta)$ is a constant depending on β, so that

$$E_n(|x|^\beta) = O(n^{-\beta}) \qquad \text{as} \quad n \to \infty.$$

In 1964, following a suggestion by H. S. Shapiro, Newman [8] proved that

$$\tfrac{1}{2} e^{-9\sqrt{n}} \leqslant R_n(|x|) \leqslant 3e^{-\sqrt{n}},$$

where $R_n(f)$ is the error of the best approximation to f on $[-1, 1]$ by rational functions of order n (that is, numerator and denominator are both polynomials of order n).

Newman's bound was improved several times by Gončar [7], Bulanov [4] and Vjačeslavov [11, 12] to

$$e^{-\pi\sqrt{n+1}} \leqslant R_n(|x|) \leqslant ce^{-\pi\sqrt{n}},$$

where c is a constant. Thus,

$$R_n(|x|) = O(e^{-\pi\sqrt{n}}) \qquad \text{as} \quad n \to \infty.$$

For $|x|^\beta$, $\beta > 0$, β not an integer, the error is

$$R_n(|x|^\beta) = O(e^{-\pi\sqrt{\beta n}})$$

(Ganelius [6]).

Rice [9] investigated the case of approximation by splines of fixed order k, but a variable number n of knots. If $S_{n,k}(f)$ denotes the error of the best approximation to f on $[-1, 1]$ by splines of order k with n knots, he showed that

$$S_{n,k}(|x|^{\beta}) \leqslant c(\beta, k) n^{-k},$$

so that

$$S_{n,k}(|x|^{\beta}) = O(e^{-k \log n}) \qquad \text{as} \quad n \to \infty.$$

For splines with a variable number of knots and variable order k_i on the ith subinterval, but fixed total order $N = \sum k_i$ the error S_N of the best approximation is (deVore and Scherer [5])

$$c_2(\beta)\, N^{-2\beta-1}(\sqrt{2}-1)^{2\sqrt{N\beta}} \leqslant S_N(|x|^{\beta}) \leqslant c_1(\beta)(\sqrt{2}-1)^{2\sqrt{N\beta}},$$

so that asymptotically

$$S_N(|x|^{\beta}) = O(e^{-c\sqrt{N} + O(\log N)}),$$

where $c = -2\sqrt{\beta}\log(\sqrt{2}-1)$.

Approximation of $|x|$ by piecewise polynomials is, of course, trivial.

Overall, we see that both $|x|$ and $|x|^{\beta}$ can be approximated with an asymptotic error of at best $O(e^{-c\sqrt{n}})$ by any set of these classical functions with n degrees of freedom.

3. Sinc Interpolation

Most of the results in this section are well known and have been summarized in Stenger [10]. We will state the relevant theorems, but include the proof of only one of them. Theorem 3.5 is not published elsewhere, and will need to refer to its proof in Section 4.

Let f be a function in $L^2(\mathbb{R})$ and analytic in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$. Whittaker's cardinal function $C(f, h)$ is defined by

$$C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\, S(k, h)(x),$$

whenever this series converges. Here h is a positive constant, and

$$S(k, h)(x) = \frac{\sin[(\pi/h)(x-kh)]}{(\pi/h)(x-kh)}.$$

The function $C(f, h)$ was first used in this form by Borel [3] in 1899, in the study of complex power series. Its use in interpolation dates back to

1915, in the work of E. T. Whittaker [13] and his son J. M. Whittaker [14, 15]. $C(f, h)$ is the unique function which interpolates f at the points kh, $k \in \mathbb{Z}$, and whose Fourier transform has support in $[-\pi/h, \pi/h]$.

The use of $C(f, h)$ and the truncated series $C_n(f, h)$ (defined below) to approximate a function f was studied by F. Stenger and others (see [10] for a survey). Many formulas can be derived by term by term integration, differentiation, Fourier transforms, Hilbert transforms, etc., along with appropriate mappings of these formulas to intervals other than $(-\infty, \infty)$. These formulas are collectively referred to as *Sinc methods*.

A class of functions for which Sinc interpolation is very accurate is defined as follows. Let D_d, $d > 0$, be the domain $D_d = \{x + iy : |y| < d\}$ in the complex plane $\mathbb{C}$ (see Fig. 1). Let $B(D_d)$ be the family of all functions which are analytic in D_d and such that

$$N(f, D_d) = \lim_{y \to d^-} \left\{ \int_{\mathbb{R}} |f(x+iy)| \, dx + \int_{\mathbb{R}} |f(x-iy)| \, dx \right\} < \infty$$

and such that

$$\int_{-d}^{d} |f(x+iy)| \, dy \to 0 \qquad \text{as} \quad x \to \pm\infty.$$

THEOREM 3.1. *If $f \in B(D_d)$, then*

$$\|f - C(f, h)\|_\infty \leqslant \frac{N(f, D_d)}{2\pi d \sinh(\pi d/h)}.$$

In practice, we have to truncate the infinite series $C(f, h)$ at some point and replace it by

$$C_n(f, h) = \sum_{k=-n}^{n} f(kh) \, S(k, h).$$

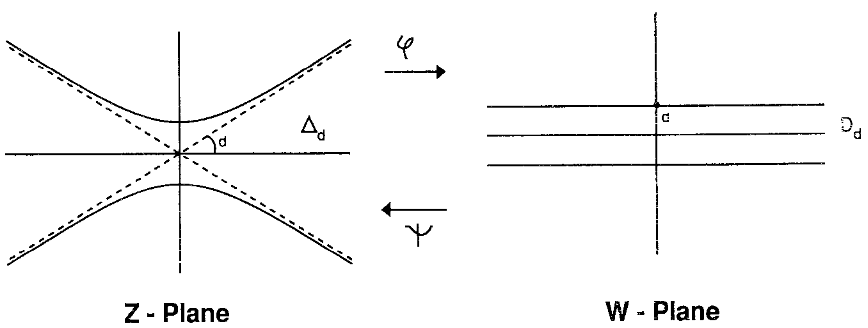

FIG. 1. The domains Δ_d and D_d.

The optimal place to do this is where the series truncation error is approximately equal to the discretization error $\|f - C(f, h)\|$. For exponentially decaying functions this leads to the following theorem.

THEOREM 3.2. *If $f \in B(D_d)$ and $|f(x)| \leqslant ce^{-\alpha|x|}$, $x \in \mathbb{R}$, where α, c are positive constants, then by choosing $h = (\pi d/\alpha n)^{1/2}$ we obtain*

$$\|f - C_n(f, h)\|_\infty \leqslant Cn^{1/2} e^{-(\pi d \alpha n)^{1/2}},$$

where C depends only on $N(f, D_d)$, d, c, α, but not on n.

Thus, we can approximate exponentially decaying functions in the class $B(D_d)$ with a uniform error which behaves asymptotically like $O(\sqrt{n}\, e^{-c\sqrt{n}})$.

Other functions with only polynomial decay at infinity can be approximated equally well if they possess a larger region of analyticity.

Let Δ_d be the domain

$$\Delta_d = \left\{ z = u + iv : \frac{v^2}{\sin^2 d} - \frac{u^2}{\cos^2 d} \leqslant 1 \right\},$$

where $0 < d < \pi/2$ (see Fig. 1).

Let $\phi(z) = \sinh^{-1} z = \log(z + \sqrt{1 + z^2})$, where we use the branches of $\sqrt{z}$, $\log(z)$ which are real on the positive real axis and slitted along the negative real axis. Let $\psi(w) = \phi^{-1}(w) = \sinh(w)$. ϕ maps Δ_d conformally onto D_d. The real axis is mapped into itself.

We can find an approximation to $f(z)$ on $\mathbb{R}$ by approximating $f(\psi(w))$ in the w-plane. This leads to

$$f(z) \approx C(f \circ \psi, h) \circ \phi(z) = \sum_{n=-\infty}^{\infty} f(z_k)\, S(k, h) \circ \phi(z),$$

where $z_k = \psi(kh) = \sinh(kh)$.

Let $B(\Delta_d)$ be the family of all functions which are analytic in Δ_d and such that

$$N(f \circ \psi, D_d) < \infty$$

and

$$\int_{-d}^{d} |f(\psi(x + iy))|\, dy \to 0 \qquad \text{as} \quad x \to \pm\infty.$$

From Theorem 3.1 we get immediately

THEOREM 3.3. *If $f \in B(\Delta_d)$, then*

$$\|f - C(f \circ \psi, h) \circ \phi\|_\infty \leqslant \frac{N(f \circ \psi, D_d)}{2\pi d \sinh(\pi d/h)}.$$

The condition

$$|f \circ \psi(x)| \leqslant c e^{-\alpha |x|}$$

is equivalent to

$$|f(x)| \leqslant c(1 + |x|)^{-\alpha},$$

and Theorem 3.2 becomes

THEOREM 3.4. *If $f \in B(\Delta_d)$ and $|f(x)| \leqslant c(1+|x|)^{-\alpha}$, $x \in \mathbb{R}$, where α, c are positive constants, then by choosing $h = (\pi d/\alpha n)^{1/2}$ we obtain*

$$\|f - C_n(f \circ \psi, h) \circ \phi\|_\infty \leqslant C n^{1/2} e^{-(\pi d \alpha n)^{1/2}},$$

where C depends on $N(f, \Delta_d)$, d, c, α, but not on n.

If it turns out that if f has both the larger region of analyticity Δ_d and exponential decay on $\mathbb{R}$, we can get a faster rate of convergence.

THEOREM 3.5. *If $f \in B(\Delta_d)$ and $|f(x)| \leqslant c e^{-\alpha |x|}$, $x \in \mathbb{R}$, where α, c are positive constants, then by choosing $h = \log n/n$, the interpolation error satisfies*

$$\|f - C_n(f \circ \psi, h) \circ \phi\|_\infty \leqslant \frac{N(f \circ \psi, D_d)}{4\pi d} e^{-\pi d n / \log n}.$$

Proof. The series truncation error T_n is

$$\begin{aligned}
T_n &= \|C(f \circ \psi, h) \circ \phi - C_n(f \circ \psi, h) \circ \phi\|_\infty \\
&\leqslant \sup_{x \in \mathbb{R}} \sum_{|k| \geqslant n+1} |F(z_k)| \, |S(k, h) \circ \sinh^{-1}(x)| \\
&\leqslant \sum_{|k| \geqslant n+1} |F(z_k)| \\
&\leqslant 2c \sum_{k=n+1}^{\infty} e^{-\alpha \sinh(kh)} \\
&\leqslant \frac{2c}{h\alpha \cosh(nh)} \int_n^\infty h\alpha \cosh(hx) e^{-\alpha \sinh(hx)} \, dx \\
&= \frac{2c}{h\alpha \cosh(nh)} e^{-\alpha \sinh(nh)},
\end{aligned}$$

while from Theorem 3.3 the discretization error E is

$$E = \|f - C(f \circ \psi, h) \circ \phi\|_\infty \leqslant \frac{1}{2\pi d \sinh(\pi d/h)} N(f \circ \sinh, D_d).$$

If $h = \log n/n$, then for large n

$$\cosh(nh) \approx \sinh(nh) \approx n/2;$$

thus asymptotically

$$T_n \leqslant \frac{4c}{\alpha \log n} e^{-\alpha n/2},$$

$$E \leqslant \frac{N(f \circ \psi, D_d)}{4\pi d} e^{-\pi d/h} = \frac{N(f \circ \psi, D_d)}{4\pi d} e^{-\pi d n/\log n}.$$

As $n \to \infty$, T_n goes to zero much faster than E, so the total error is asymptotically bounded by E. ▌

4. Application to Approximation Theory

Let us now consider the approximation of the function $|x|^\beta$, $\beta > 0$, on $[-1, 1]$. (The case $\beta = 1$ requires no special treatment). Through the transformation

$$z = \cosh^{-1}\left(\frac{1}{|x|}\right) \operatorname{sign}(x) \leftrightarrow x = (1/\cosh(z)) \operatorname{sign}(z)$$

this is equivalent to the approximation of $f(z) = 1/\cosh^\beta(z)$ on the real line. f is analytic in the complex plane slitted from i to $i\infty$ and from $-i$ to $-i\infty$, and decays like $e^{-\beta|x|}$ on the real line.

The interpolation formula reads in this case

$$\frac{1}{\cosh^\beta z} \approx \sum_{k=-n}^{n} \frac{1}{\cosh^\beta(\sinh(kh))} S(k, h) \circ \sinh^{-1}(z)$$

or

$$\begin{aligned} |x|^\beta &\approx \sum_{k=-n}^{n} \frac{1}{\cosh^\beta(\sinh(kh))} S(k, h) \circ \sinh^{-1}\left(\cosh^{-1}\left(\frac{1}{|x|}\right) \operatorname{sign}(x)\right) \\ &= \sum_{k=-n}^{n} \frac{1}{\cosh^\beta(\sinh(kh))} S(k, h) \circ \sinh^{-1}\left(\cosh^{-1}\left(\frac{1}{|x|}\right)\right). \end{aligned}$$

We will now show that $f(z) = 1/\cosh^{\beta}(z)$ satisfies the conditions of Theorem 3.5 with $d = 1$.

From the identity

$$|\cosh(\sinh(x+iy))|^2 = \cosh^2[\sinh x \cos y] \cos^2[\cosh x \sin y] + \sinh^2[\sinh x \cos y] \sin^2[\cosh x \sin y] \geqslant \sinh^2[\sinh x \cos y]$$

we see than on the lines $\{x \pm i : x \in \mathbb{R}\}$ the absolute value of the function $g(w) = 1/\cosh(\sinh(w))$ is greater than 1, but bounded above, for $w = x \pm i$, x near 0, and decreases exponentially for large $|x|$. Given any $0 < \beta_0 < \beta_1$, we can estimate $|g(w)|^{\beta}$ by $|g(w)|^{\beta_1}$, where $|g(w)| > 1$, and by $|g(w)|^{\beta_0}$, where $|g(w)| \leqslant 1$, to find that

$$N\left(\frac{1}{\cosh^{\beta}(\sinh z)}, D_1\right) \leqslant M$$

uniformly for all $0 < \beta_0 \leqslant \beta \leqslant \beta_1$.

Let $E_n(|x|^{\beta})$ be the error of the uniform best approximation to $|x|^{\beta}$ on $[-1, 1]$ by linear combinations of the basis functions

$$B_{n,k}(x) = S(k, h) \circ \sinh^{-1}\left(\cosh^{-1}\left(\frac{1}{|x|}\right)\right),$$

where $h = \log n/n$, $k = -n, \ldots, n$.

The proof of Theorem 3.5 shows that asymptotically

$$E_n(|x|^{\beta}) \leqslant \frac{M}{4\pi} e^{-\pi n/\log n},$$

uniformly for $0 < \beta_0 \leqslant \beta \leqslant \beta_1$.

References

1. S. N. Bernstein, Sur la valeur asymptotique de la meilleure approximation de $|x|$, *Acta Math.* **37** (1913), 1–57.
2. S. N. Bernstein, Sur la meilleure approximation de $|x|^p$ par des polynômes de degrées très élevés, *Bull. Acad. Sci. USSR, Cl. Sci. Math. Nat.* **2** (1938), 181–190.
3. E. Borel, Mémoire sur les séries divergentes, *Ann. Ecole Norm. Sup. (3)* **16** (1899), 9–131.
4. A. P. Bulanov, Asymptotics for least deviation of $|x|$ from rational functions, *Math. USSR Sb.* **5** (1968), 275–290. [Russian original *Mat. Sb.* **76** (118), 288–303.]
5. R. A. DeVore and K. Scherer, Variable knot, variable degree spline approximation to x^{β}, *in* "Quantitative Approximation" (R. A. DeVore and K. Scherer, Eds.), pp. 121–132, Academic Press, New York, 1980.

6. T. Ganelius, Rational approximation to x^α on [0, 1], *Anal. Math.* **5** (1979), 19–33.
7. A. A. Gončar. Estimates of the growth of rational functions and some of their applications, *Math. USSR Sb.* **1** (1967), 445–456. [Russian original *Mat. Sb.* **72** (114), 489–503.]
8. D. J. Newman, Rational approximation to $|x|$, *Michigan Math. J* **11** (1964), 11–14.
9. J. R. Rice, On the degree of convergence of nonlinear spline approximation, *in* "Approximations with Special Emphasis on Spline Functions" (I. J. Schoenberg, Ed.), pp. 349–365, Academic Press, New York, 1969.
10. F. Stenger, Numerical methods based on Whittaker cardinal or sinc functions. *SIAM Rev.* **23** (1981). 165–224.
11. N. S. Vjačeslavov, Approximation of the function $|x|$ by rational functions, *Math. Notes* **16** (1974), 680–685. [Russian original *Mat. Zametki* **16** (1974), 163–171.)
12. N. S. Vjačeslavov, On the uniform approximation of $|x|$ by rational functions, *Soviet Math. Dokl.* **16** (1975), 100–104. [Russian original *Dokl. Akad. Nauk SSSR* **220** (1975), 512–515.]
13. E. T. Whittaker, On the functions which are represented by the expansions of the interpolation theory, *Proc. Roy. Soc. Edinburgh* **35** (1915), 181–194.
14. J. M. Whittaker, On the cardinal function of interpolation theory, *Proc. Edinburgh Soc. Ser. 1* **2** (1927), 41–46.
15. J. M. Whittaker, "Interpolatory Function Theory," Cambridge Univ. Press, London, 1935.